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NORMALIZING CONNECTIONS AND THE ENERGY-MOMENTUM METHOD

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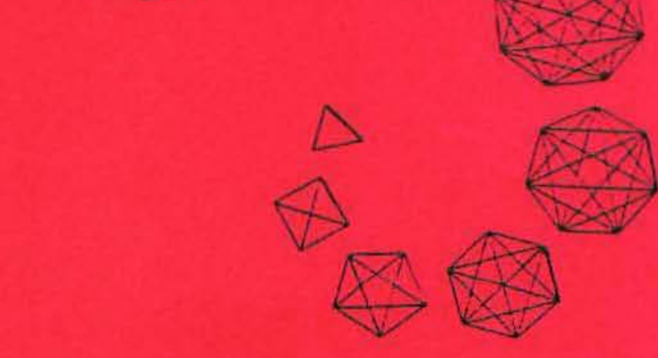

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Abstract

The block diagonalization method for determining the stability of relative equilibria is discussed from the point of view of connections. We construct connections whose horizontal and vertical decompositions *simultaneously* put the second variation of the augmented Hamiltonian and the symplectic structure into normal form. The cotangent bundle reduction theorem provides the setting in which the results are obtained.

Introduction In Simo, Posbergh and Marsden [1990a,b], Lewis and Simo [1990] and Simo, Lewis and Marsden [1990], a powerful method for determining the stability of relative equilibria in Hamiltonian systems with symmetry is developed. The technique provides useful tools in the emerging theory of bifurcation of relative equilibria. The main examples treated in these works are rotating systems, such as rotating elastic bodies and rigid bodies with flexible attachments. Coupled rigid bodies are treated in Patrick [1990].

The main feature of these results is a splitting of the space of variations that simultaneously puts the second variation of the augmented Hamiltonian and the symplectic form into a normal form. This splitting is defined by explicit formulas that allow one to readily implement the conditions in complex examples.

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The purpose of this paper is not to pursue the applications of the above ***normal form method*** (or ***block-diagonalizaton method***), but rather to explore its intrinsic geometry. The splittings of variables giving the normal form will be shown to be the horizontal-vertical splittings of connections associated with the cotangent bundle reduction theorem. The connections are Ehresmann connections, but as we show, they are affiliated to a large extent with principal bundle connections in the standard sense. One of the connections is the mechanical connection implicitly found in Smale [1970] and used in Guichardet [1984] and Marsden, Montgomery and Ratiu [1990] to study geometric phases (Cartan-Hannay-Berry phases). This latter reference also contains additional relevant references. The mechanical connection occurs in the bundle whose base space is the reduced space itself—in the present work a different and much less obvious connection is needed on a bundle over shape space. (It is only in the abelian case that the two bundles are coincident). These two connections interweave to produce connections on a variety of related bundles and ultimately yield the normal forms.

Another comment is important here. It is clear in advance that splittings block diagonalizing the second variation of the augmented energy must exist—one can use the second variation as a "metric" and take the orthogonal complement of shape space. However, what is remarkable is that *the splittings can be given so explicitly,* in advance of the computation of the second variation and secondly, that *they simultaneously bring the symplectic form into normal form.* In particular, this means that the linearized equations of motion are brought into normal form.

There are a number of important questions that warrant future investigation. Among them are the calculations of the curvatures of the connections used here and their role in stability theory. Another is the use of these splittings in optimal control. In Montgomery [1990], it is shown how optimal control strategies are governed by solutions of the equations of motion of a particle in a Yang-Mills field—the field being the mechanical connection. However, this is done in a context (such as zero total angular momenum) where parts of the splitting collapse and the full power of block diagonalization cannot be utilized. In fact, block diagonalization was designed to deal with situations in which the total angular momentum is not zero, such as rotating gravitating bodies, rotating satellites and rotating molecules. We believe that these two lines of investigation can be brought into a closer union.

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§1 *The Energy-Momentum Method*

This section provides a concise summary of the energy-momentum method following the references cited in the introduction. The energy-momentum method itself first appeared in Marsden, Lewis, Posbergh and Simo [1989] and Simo, Posbergh and Marsden [1990a]. Its development was motivated by the energy-Casimir method (see Holm, Marsden, Ratiu and Weinstein [1985] and references therein), which "fails" due to the lack of an adequate number of Casimir invariants in some key examples in elasticity and fluid dynamics, such as the theory of rods. The energy momentum technique can be traced back to Riemann and Poincaré in specific examples and may be regarded as implicit in Arnold [1966] and Marsden and Weinstein [1974].

Let (P, Ω) be a symplectic manifold, G a Lie group acting by symplectic transformations and let $\mathfrak{g}$ denote its Lie algebra and $\mathfrak{g}^*$ the vector space dual of $\mathfrak{g}$. Let $J : P \to \mathfrak{g}^*$ be an equivariant momentum map for the action and let $H : P \to R$ be a G-invariant Hamiltonian. In later sections, we specialize to the case of cotangent bundles, but for now we work with general symplectic manifolds. Let X_H be the Hamiltonian vector field of H and let F_t^H be its flow. Recall that by definition of the momentum map, $X_{\langle J,\xi\rangle} = \xi_P$ is the infinitesimal action of $\mathfrak{g}$ on P; equivalently, $F_t^{\langle J,\xi\rangle}(z) = \exp(t\xi)z$ for $z \in P$, where gz denotes the action of $g \in G$ on z. Here and below, $\langle\, ,\rangle$ denotes the natural pairing.

A ***relative equilibrium*** is a point $z_e \in G$ such that $F_t^H(z_e) = \exp(t\xi_e)z_e$ for some $\xi_e \in \mathfrak{g}$; in other words, *a relative equilibrium is a phase space point whose dynamic orbit is also a one-parameter group orbit.* Conservation and equivariance of J imply that $\xi_e \in \mathfrak{g}_{\mu_e}$ where $\mu_e = J(z_e)$ is the value of J at the relative equilibrium z_e and $\mathfrak{g}_{\mu_e}$ is the coadjoint isotropy Lie algebra of μ_e.

There are a number of equivalent characterizations of relative equilibria, but the one most relevant at the moment is as follows.

1.1 Relative Equilibrium Theorem *A point $z_e \in P$ is a relative equilibrium iff there is a $\xi_e \in \mathfrak{g}$ such that z_e is a critical point of the augmented Hamiltonian*

$$H_{\xi_e} := H - \langle J - \mu_e, \xi_e\rangle \tag{1}$$

This theorem follows from differentiation of the defining condition $F_t^H(z_e) = \exp(t\xi_e)z_e$ with respect to t at $t = 0$.

If μ_e is a regular value of J (*i.e.*, if the isotropy algebra $\mathfrak{g}_{z_e}$ of z_e is $\{0\}$), then z_e is a relative equilibrium iff $H \mid J^{-1}(\mu_e)$ has a critical point at z_e. In fact, **1.1** may be viewed as a version of the Lagrange multiplier theorem, where ξ_e is the multiplier.

For theorem 1.2 and the further developments below, we assume that μ_e is a regular value of $\mathbf{J}$ and that the action of G on P is free and proper. We also assume that either μ_e is a generic point in $\mathfrak{g}^*$ or that the adjoint action of G_{μ_e} on $\mathfrak{g}$ admits an invariant inner product.

We refer to Patrick [1990] for more details and examples concerning these assumptions. An example showing that μ_e being a regular value and the action of G on P being free and proper is not enough, is discussed in Krishnaprasad [1989]. Of considerable interest in the theory of bifurcation with symmetry is to investigate what happens when these assumptions fail, especially the assumption that μ_e is not regular; *i.e.*, z_e has nontrivial isotropy (*i.e.*, symmetry) algebra.

The ***energy-momentum method*** proceeds as follows. Choose a linear subspace $\mathcal{S} \subset T_{z_e}J^{-1}(\mu_e) = \ker T_{z_e}J(z_e)$ that is transverse to the G_{μ_e}-orbit of z_e, where G_{μ_e} is the coadjoint isotropy subgroup of μ_e. Note that $\mathcal{S}$ is isomorphic to the tangent space to the reduced manifold $P_{\mu_e} = J^{-1}(\mu_e)/G_{\mu_e}$ (since the G_{μ_e} action is assumed to be free and proper, the quotient is a manifold).

1.2 Theorem *If $\delta^2 H_{\xi_e}(z_e)$, the second variation of H_{ξ_e} evaluated at z_e, is positive or negative definite when restricted to $\mathcal{S}$, then z_e is G_{μ_e}-orbitally stable. That is, z_e is dynamically stable mod the G_{μ_e}-action on P.*

We note that $\delta^2 H_{\xi_e}(z_e) \mid \mathcal{S}$ projects to the second variation of the reduced Hamiltonian on P_{μ_e} at $\pi_{\mu_e}(z_e)$, the projection of z_e; definiteness of this reduced second variation is a well-known sufficient condition for stability.

Now we turn our attention to the case of cotangent bundles. Thus, assume $P = T^*Q$ with its canonical symplectic structure, G acts by cotangent lift and $\mathbf{J}$ is the standard momentum map for this case. Assume that Q carries a Riemannian metric, denoted $\langle\langle\, , \rangle\rangle$ and that G acts on Q by isometries. Finally, assume H is of the form $K + V$ where K is the kinetic energy of the metric $\langle\langle\, , \rangle\rangle$ and $V : Q \to \mathbb{R}$ is a given G-invariant potential. We choose this context to be explicit but we note that much of what is done here generalizes to a large class of G-invariant Lagrangians $L : TQ \to \mathbb{R}$ (see Lewis [1990]).

Define, for each $q \in Q$, the ***locked inertia tensor*** $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ by

$$\langle \mathbb{I}(q)\,\xi, \eta \rangle = \langle\langle \xi_Q(q), \eta_Q(q) \rangle\rangle\,, \tag{2}$$

where ξ_Q denotes the infinitesimal generator of $\xi \in \mathfrak{g}$. Note that $\mathbb{I}(q)$ is symmetric relative to the natural pairing $\langle\, ,\rangle$. Since we have a free action, $\mathbb{I}(q)$ is invertible. Define

$$\alpha : TQ \to \mathfrak{g} \qquad \text{by} \qquad \alpha(v_q) = \mathbb{I}(q)^{-1}\mathbf{J}(\mathbb{F}L(v_q)) \tag{3}$$

where $\mathbb{F}L(v_q)$ is the momentum associated to the velocity v_q via the Legendre transform. Thus, α gives the "angular velocity of the locked system". We will use the terms "angular velocity" and "angular momentum" for elements of $\mathfrak{g}$ and $\mathfrak{g}^*$, even though G need not be $SO(3)$.

1.3 Proposition *The locked angular velocity α defines a connection on the principal G-bundle $Q \to S = Q/G$, if the G action is free and proper. We call α **the mechanical connection.***

Proof The vertical subspace of the G-bundle $Q \to S$ at a point q is the tangent space to the group orbit, $G\cdot q$, so the vertical vectors are the infinitesimal generators $\xi_Q(q)$. The first step in verifying that α is a connection is to check that $\alpha(\xi_Q(q)) = \xi$, which can be seen as follows. Let $p_q \in T_q^*Q$. From $\langle \mathbf{J}(p_q), \xi\rangle = \langle p_q, \xi_Q(q)\rangle$, we obtain

$$\begin{aligned}\langle \nu, \alpha(\xi_Q(q))\rangle &= \langle \nu, \mathbb{I}(q)^{-1}\mathbf{J}(\mathbb{F}L(\xi_Q(q)))\rangle = \langle \mathbb{I}(q)^{-1}\nu, \mathbf{J}(\mathbb{F}L(\xi_Q(q))\rangle \\ &= \langle \mathbb{F}L(\xi_Q(q)), (\mathbb{I}(q)^{-1}\nu)_Q(q)\rangle = \langle \nu, \xi\rangle,\end{aligned}$$

for any $\nu \in \mathfrak{g}^*$, and so

$$\alpha(\xi_Q(q)) = \xi.$$

The second step is to check that α *is equivariant* in the sense that $\alpha(T_q\Phi_h\cdot v_q) = \mathrm{Ad}_h\alpha(v_q)$, where $\Phi_h(q) = h\cdot q$ denotes the left action of h on Q. To show this, we make use of G-invariance of the metric, equivariance of the momentum map $\mathbf{J}$, and the following equivariance property of the locked inertia tensor

$$\mathrm{Ad}_h^*(\mathbb{I}(h\cdot q)\mathrm{Ad}_h\zeta) = \mathbb{I}(q)\zeta. \tag{4}$$

To prove (4), let $h \in G$, $q \in Q$, and $\eta, \zeta \in \mathfrak{g}$. Using G-invariance of the metric,

$$\begin{aligned}\langle \mathrm{Ad}_h^*(\mathbb{I}(h\cdot q)\mathrm{Ad}_h\zeta), \eta\rangle &= \langle \mathbb{I}(h\cdot q)\mathrm{Ad}_h\zeta, \mathrm{Ad}_h\eta\rangle = \langle\langle (\mathrm{Ad}_h\eta)_Q(h\cdot q), (\mathrm{Ad}_h\zeta)_Q(h\cdot q)\rangle\rangle_{h\cdot q} \\ &= \langle\langle T_q\Phi_h\cdot\eta_Q(q), T_q\Phi_h\cdot\zeta_Q(q)\rangle\rangle_{h\cdot q} = \langle\langle \eta_Q(q), \zeta_Q(q)\rangle\rangle_q = \langle \mathbb{I}(q)\zeta, \nu\rangle.\end{aligned}$$

Inverting (4), we obtain

$$\mathrm{Ad}_{h^{-1}} \circ \mathbb{I}(h\cdot q)^{-1} \circ \mathrm{Ad}^*_{h^{-1}} = \mathbb{I}(q)^{-1}.$$

Therefore,

$$\alpha(T_q\Phi_h\cdot v_q) = (\mathbb{I}(h\cdot q)^{-1} \circ \mathbf{J} \circ \mathbb{F}L)(T_q\Phi_h\cdot v_q) = \mathbb{I}(h\cdot q)^{-1}[\mathbf{J}(T_q^*\Phi_{h^{-1}}(\mathbb{F}L(v_q)))]$$

$$= \mathbb{I}(h\cdot q)^{-1}[\mathrm{Ad}^*_{h^{-1}}(\mathbf{J}(\mathbb{F}L(v_q))] = \mathrm{Ad}_h\cdot \mathbb{I}(q)^{-1}[\mathbf{J}(\mathbb{F}L(v_q))] = \mathrm{Ad}_h(\alpha(v_q)).$$

Thus α is G-equivariant and hence is a connection. ■

Another useful observation is that α *is the connection defined by choosing the horizontal spaces to be* $\langle\langle\, ,\rangle\rangle$*-orthogonal to the* G*-orbits*; *i.e.*, the spaces of vectors with zero "angular" momentum. Defining $\alpha_\mu : Q \to T^*Q$ by

$$\langle \alpha_\mu(q), v_q\rangle = \langle \mu, \alpha(v_q)\rangle, \tag{5a}$$

we observe that α_μ takes values in $\mathbf{J}^{-1}(\mu)$, a fact valid for any connection; indeed,

$$\langle \mathbf{J}(\alpha_\mu(q)), \eta\rangle = \langle \alpha_\mu(q), \eta_Q(q))\rangle = \langle \mu, \alpha(\eta_Q(q))\rangle = \langle \mu, \eta\rangle,$$

which follows from the proof of Proposition 1.3.

Another useful observation is that if $\zeta = \mathbb{I}(q)^{-1}\mu$, then

$$\alpha_\mu(q) = \mathbb{F}L(\zeta_Q(q)), \tag{5b}$$

as is readily verified.

The ***amended potential*** V_μ is defined for each $\mu \in \mathfrak{g}^*$ to be

$$V_\mu(q) = H(\alpha_\mu(q)), \tag{6a}$$

as in Smale [1970] and Abraham and Marsden [1978]. One finds from the definitions that

$$V_\mu(q) = V(q) + \tfrac{1}{2}\langle \mu, \mathbb{I}(q)^{-1}\mu\rangle\ . \tag{6b}$$

Here is the verification of (6b); we start with (6a) and compute with the aid of (5b) as follows:

$$H(\alpha_\mu(q)) = \tfrac{1}{2}\langle\langle \alpha_\mu(q), \alpha_\mu(q)\rangle\rangle + V(q) = \tfrac{1}{2}\langle\langle \mathbb{F}L(\zeta_Q(q)), \mathbb{F}L(\zeta_Q(q))\rangle\rangle + V(q)$$

$$= \frac{1}{2}\langle \mathbb{I}(q)\zeta, \zeta\rangle + V(q) = \frac{1}{2}\langle \mu, \mathbb{I}(q)^{-1}\mu\rangle + V(q) = V_\mu(q).$$

For $p_q \in T_q^*Q$, define

$$K_\mu(p_q) = \frac{1}{2}\| p_q - \alpha_\mu(q) \|^2 \tag{6c}$$

and let the functions V_ξ and K_ξ be defined by

$$V_\xi(q) = V(q) - \frac{1}{2}\langle \xi, \mathbb{I}(q)\xi\rangle \tag{7a}$$

and

$$K_\xi(p_q) = \frac{1}{2}\| p_q - \mathbb{F}L\cdot\xi_Q(q) \|^2. \tag{7b}$$

Fix $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$ and let, as in equation (1),

$$H_{\xi,\mu} = H - \langle J - \mu, \xi\rangle. \tag{8}$$

Then one has the following easily verified identities:

$$H_{\xi,\mu} = K_\xi + V_\xi + \langle \mu, \xi\rangle \tag{9}$$

and

$$K_\mu + V_\mu = H - \langle J - \mu, \xi\rangle, \tag{10}$$

where

$$\xi(q) = \mathbb{I}(q)^{-1}\mu .$$

To assist in what follows, one can first verify that at a relative equilibrium corresponding to ξ_e (see **1.1**), one has the identities

$$\xi_e = \mathbb{I}(q_e)^{-1}\mu_e \tag{11}$$

and (see (5b) above)

$$p_e = \alpha_{\mu_e}(q_e) = \mathbb{F}L((\xi_e)_Q(q_e)) \tag{12}$$

1.4 Reduced Relative Equilibrium Theorem *The following are equivalent for a point* $z_e \in T_{q_e}^*Q$:

- **i** z_e *is a relative equilibrium*
- **ii** *there is a* $\xi_e \in \mathfrak{g}$ *such that* $z_e = \mathbb{F}L((\xi_e)_Q(q_e))$ *and* q_e *is a critical point of* V_{ξ_e}
- **iii** $z_e = \alpha_{\mu_e}(q_e)$ *and* q_e *is a critical point of* V_{μ_e}, *where* $\mu_e = J(z_e)$
- **iv** z_e *is a critical point of* $H \mid J^{-1}(\mu_e)$.

Proof We already noted the equivalence **i** $\Leftrightarrow$ **iv** in Theorem **1.1**. The equivalence **i** $\Leftrightarrow$ **ii** follows from Theorem **1.1** and equation (9), and the equivalence **iii** $\Leftrightarrow$ **iv** follows from equation (10). ■

In condition **iii**, the ξ_e in the definition of relative equilibrium is given by equation (11). We also observe that $K_\mu + V_\mu$ has the correct form of the reduced hamiltonian on P_μ. This is consistent with the cotangent bundle reduction theorem described in the next section, and with the fact that it is $\delta^2 V_\mu$ on the appropriate space that gives the reduced energy-momentum method.

§2 *The Cotangent Bundle Reduction Theorem*

The cotangent bundle reduction theorem of Satzer, Marsden and Kummer gives a realization of the reduced space $P_\mu = J^{-1}(\mu)/G$ in case $P = T^*Q$. The following diagram summarizes the situation:

$$\begin{tikzcd}
T^*Q \arrow[ddd] & \supset & J^{-1}(\mu) \arrow[d, "G_\mu"] & \subset & J^{-1}(\mathcal{O}) \arrow[d, "G"'] & \subset & T^*Q \arrow[ddd] \\
 & & P_\mu \arrow[d, "\text{injection}"'] & \cong & P_{\mathcal{O}} \arrow[d, "\text{surjection}", "\mathcal{O}"'] & & \\
 & & T^*(Q/G_\mu) \arrow[d] & & T^*(Q/G) \arrow[d] & & \\
Q \arrow[rr] & & Q/G_\mu \arrow[rr] & & Q/G & & Q \arrow[ll]
\end{tikzcd}$$

One version of the theorem (see Abraham and Marsden [1978]) says that P_μ *embeds* as a vector subbundle of $T^*(Q/G_\mu)$—this is the injection in the above figure. Another version (see Marle [1976], Kazhdan, Kostant and Sternberg [1978] and Marsden [1981]) says that $P_\mu \cong P_{\mathcal{O}}$ is a *coadjoint bundle* over $T^*(Q/G)$ with fiber the coadjoint orbit $\mathcal{O}$ through μ. Both are proved by a similar technique. We state it as follows:

2.1 Cotangent Bundle Reduction P_μ *is a bundle over* $T^*(Q/G)$ *with fiber* $\mathcal{O}$.

Step 1 Reduction at zero

$$(T^*Q)_0 \cong T^*(Q/G). \tag{1}$$

Here $(T^*Q)_0 = J^{-1}(0)/G$, since $G_0 = G$. We can identify this quotient with $T^*(Q/G)$ by observing that $\beta_q \in J^{-1}(0)$ if and only if $\beta_q \in [\mathfrak{g}\cdot q]^A$, *i.e.*, $\langle \beta_q, \eta_Q(q)\rangle = 0$ for all $\eta \in \mathfrak{g}$; hence the pairing of β_q with vectors modulo generators $\eta_Q(q)$ is well-defined and we can regard β_q as an element of $T^*_{[q]}(Q/G)$. The symplectic form on $T^*(Q/G)$ is the canonical one. (See Abraham and Marsden [1978].)

Step 2 Orbit reduction *The reduced space* $(T^*Q)_\mu = J^{-1}(\mu)/G_\mu$ *can be identified with the quotient* $J^{-1}(\mathcal{O})/G$, *where* $\mathcal{O}$ *is the coadjoint orbit through* μ. (See Marsden [1981], Marsden, Weinstein, Ratiu, Schmidt, and Spencer [1983] and references therein.)

Step 3 Shifting *Use the shift map* Σ *defined by* $\Sigma(z) = z - \alpha_{J(z)}(q)$ (*where* $z \in T^*_qQ$) *to map* $J^{-1}(\mathcal{O})$ to $J^{-1}(0)$.

Letting $\alpha_z \in T^*_qQ$ be defined by $\alpha_z = \alpha_{J(z)}(q)$, we can write $\Sigma(z) = z - \alpha_z$. We claim Σ is equivariant with respect to the G-action. To see this, let $h \in G$ and note that

$$\begin{aligned}\langle T^*_q\Phi_h\cdot\alpha_{h\cdot z}, v_q\rangle &= \langle \alpha_{h\cdot z}, T_q\Phi_h\cdot v_q\rangle = \langle J(h\cdot z), \alpha(T_q\Phi_h\cdot v_q)\rangle \\ &= \langle \mathrm{Ad}^*_{h^{-1}}J(z), \mathrm{Ad}_h\alpha(v_q)\rangle = \langle \alpha_z, v_q\rangle .\end{aligned}$$

Thus, $T^*_q\Phi_h\cdot\alpha_{h\cdot z} = \alpha_z$ and so Σ is equivariant and hence drops to the quotient, producing the desired map

$$(T^*Q)_\mu = J^{-1}(\mathcal{O})/G \to J^{-1}(0)/G = T^*(Q/G). \tag{2}$$

This map has fiber $\mathcal{O}$, *i.e.*, $\Sigma \circ \alpha_\mu = 0$ for all $\mu \in \mathcal{O}$, so our assertion is proved. ■

Below, we show how the canonical symplectic structure restricted to the subspace $\mathcal{S}$ can be identified with the Poisson structure on the bundle $P_{\mu_e} \to Q/G$. The calculation of the Poisson structure for general reductions of principal bundles is given in Marsden, Montgomery, and Ratiu

[1984]. (See also Lewis, Marsden, and Ratiu [1987] for an application to the dynamics of systems with free boundaries.)

§3 *The Rigid-Internal Splitting*

We now describe the splitting of the space of variations in the energy-momentum method. First, fix a point $z = p_q \in T_q^*Q$ and fix $\mu = J(p_q)$. Define

$$\mathcal{V} = \{\delta q \in T_qQ \mid \langle\langle \delta q, \zeta_Q(q) \rangle\rangle = 0 \text{ for all } \zeta \in \mathfrak{g}_\mu\} \tag{1a}$$

and let

$$\mathcal{S} = \{\delta z = (\delta q, \delta p) \in T_z(T^*Q) \mid T_zJ(z)\cdot\delta z = 0 \text{ and } \delta q \in \mathcal{V}\} \tag{1b}$$

This is the choice of $\mathcal{S}$ taken in the energy-momentum method—other choices are also possible, but we choose (1b) in this paper for definiteness.

Next, we split $\mathcal{V} = \mathcal{V}_{\mathrm{RIG}} \oplus \mathcal{V}_{\mathrm{INT}}$ in the following way. Define

$$\mathcal{V}_{\mathrm{RIG}} = \{\eta_Q(q) \in T_qQ \mid \eta \in \mathfrak{g}_\mu^\perp\} \tag{2}$$

where $\mathfrak{g}_\mu^\perp$ is the orthogonal complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$ with the locked inertia metric. (This choice of orthogonal complement depends on q, but we do not include this in the notation). From (1a) it is clear that $\mathcal{V}_{\mathrm{RIG}} \subset \mathcal{V}$ and that $\mathcal{V}_{\mathrm{RIG}}$ has the dimension of the coadjoint orbit through μ. Next, define

$$\mathcal{V}_{\mathrm{INT}} = \{\delta q \in \mathcal{V} \mid \langle \eta, [D\mathbb{I}(q)\cdot\delta q]\cdot\xi \rangle = 0 \text{ for all } \eta \in \mathfrak{g}_\mu^\perp\} \tag{3a}$$

where $\xi = \mathbb{I}(q)^{-1}\mu$. An equivalent definition is

$$\mathcal{V}_{\mathrm{INT}} = \{\delta q \in \mathcal{V} \mid [D\mathbb{I}(q)^{-1}\cdot\delta q]\cdot\mu \in \mathfrak{g}_\mu\}, \tag{3b}$$

which is clear from (3a). The definition of $\mathcal{V}_{\mathrm{INT}}$ has an important mechanical interpretation in terms of the objectivity of the centrifugal force in case $G = SO(3)$; see Simo, Lewis and Marsden [1990] and Simo, Posbergh and Marsden [1990a,b].

Define the ***Arnold form*** $\mathcal{A}_\mu : \mathfrak{g}_\mu^\perp \times \mathfrak{g}_\mu^\perp \to \mathbb{R}$ by

$$\mathcal{A}_\mu(\eta, \zeta) = \langle \mathrm{ad}_\eta^*\mu, \chi_{(q,\mu)}(\zeta) \rangle = \langle \mu, \mathrm{ad}_\eta \chi_{(q,\mu)}(\zeta) \rangle, \tag{4a}$$

where $\chi_{(q,\mu)} : \mathfrak{g}_\mu^\perp \to \mathfrak{g}$ is defined by

$$\chi_{(q,\mu)}(\zeta) = \mathbb{I}(q)^{-1}\mathrm{ad}_\zeta^*\mu + \mathrm{ad}_\zeta\mathbb{I}(q)^{-1}\mu \tag{4b}$$

The Arnold form appears in Arnold's [1966] stability analysis of relative equilibria in the special case $Q = G$. At a relative equilibrium, the form $\mathcal{A}_\mu$ is symmetric, as is verified either directly or by recognizing it as the second variation of V_ξ on $\mathcal{V}_{RIG} \times \mathcal{V}_{RIG}$ (see Simo, Lewis and Marsden [1990] for details). At a relative equilibrium, the form $\mathcal{A}_\mu$ is degenerate as a symmetric bilinear form on $\mathfrak{g}_\mu^\perp$ when there is a non-zero $\zeta \in \mathfrak{g}_\mu^\perp$ such that

$$\mathbb{I}(q)^{-1}\mathrm{ad}_\zeta^*\mu + \mathrm{ad}_\zeta\mathbb{I}(q)^{-1}\mu \in \mathfrak{g}_\mu\,; \tag{4c}$$

in other words, when $\mathbb{I}(q)^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ has a *nontrivial symmetry* relative to the coadjoint-adjoint action of $\mathfrak{g}$ (restricted to $\mathfrak{g}_\mu^\perp$) on the space of linear maps from $\mathfrak{g}^*$ to $\mathfrak{g}$. (When one is not at a relative equilibrium, we say the Arnold form is *non-degenerate* when $\mathcal{A}_\mu(\eta, \zeta) = 0$ for all $\eta \in \mathfrak{g}_\mu^\perp$ implies $\zeta = 0$.) This means, for $G = \mathbf{SO}(3)$ that $\mathcal{A}_\mu$ is non-degenerate if μ is not in a multidimensional eigenspace of I^{-1}. Thus, *if the locked body is not symmetric (i.e., a Lagrange top), then the Arnold form is non-degenerate.*

3.1 Proposition *If the Arnold form is non-degenerate, then*

$$\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT} \tag{5}$$

Indeed, non-degeneracy of the Arnold form implies $\mathcal{V}_{RIG} \cap \mathcal{V}_{INT} = \{0\}$ and, at least in the finite dimensional case, a dimension count gives **3.1**. In the infinite dimensional case, the relevant ellipticity conditions are neded.

The split (5) can now be used to induce a split of the phase space

$$\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}\,. \tag{6}$$

Using a more mechanical viewpoint, Simo, Lewis and Marsden [1990] show how $\mathcal{S}_{RIG}$ can be defined by extending $\mathcal{V}_{RIG}$ from positions to momenta using *superposed rigid motions*. For our purposes, the important characterization of $\mathcal{S}_{RIG}$ is

$$\mathcal{S}_{RIG} = T_q\alpha_\mu \cdot \mathcal{V}_{RIG} \tag{7}$$

so $\mathcal{S}_{RIG}$ is isomorphic to $\mathcal{V}_{RIG}$. Since α_μ maps Q to $J^{-1}(\mu)$ and $\mathcal{V}_{RIG} \subset \mathcal{V}$, we get $\mathcal{S}_{RIG} \subset \mathcal{S}$. Define

$$\mathcal{S}_{\mathrm{INT}} = \{\delta z \in \mathcal{S} \mid \delta q \in \mathcal{V}_{\mathrm{INT}}\}; \tag{8}$$

(6) holds if the Arnold form is non-degenerate. Next, we write

$$\mathcal{S}_{\mathrm{INT}} = \mathcal{W}_{\mathrm{INT}} \oplus \mathcal{W}^*_{\mathrm{INT}}, \tag{9}$$

where $\mathcal{W}_{\mathrm{INT}}$ and $\mathcal{W}^*_{\mathrm{INT}}$ are defined as follows:

$$\mathcal{W}_{\mathrm{INT}} = T_q\alpha_\mu \cdot \mathcal{V}_{\mathrm{INT}} \tag{10a}$$

and

$$\mathcal{W}^*_{\mathrm{INT}} = \{\upsilon(\gamma) \mid \gamma \in [\mathfrak{g}\cdot q]^A\} \tag{10b}$$

where $\mathfrak{g}\cdot q = \{\zeta_Q(q) \mid \zeta \in \mathfrak{g}\}$, $[\mathfrak{g}\cdot q]^A \subset T^*_qQ$ is its annihilator, and $\upsilon(\gamma) \in T_z(T^*Q)$ is the vertical lift of $\gamma \in T^*_qQ$; in coordinates, $\upsilon(q^i, \gamma_j) = (q^i, p_j, 0, \gamma_j)$. The vertical lift is given intrinsically by taking the tangent to the curve $\sigma(s) = z + s\gamma$ at $s = 0$.

§4 *The Normal Form*

4.1 Block Diagonalization Theorem *In the splittings introduced in* §3 *and at a relative equilibrium,* $\delta^2 H_\xi(z_e)$ *and the symplectic form* Ω_{z_e} *have the form:*

$$\delta^2 H_\xi(z_e) = \begin{array}{c} \begin{array}{ccc} \mathcal{S}_{\mathrm{RIG}} & \mathcal{W}_{\mathrm{INT}} & \mathcal{W}^*_{\mathrm{INT}} \end{array} \\ \begin{bmatrix} \begin{bmatrix} \text{Arnold} \\ \text{form} \end{bmatrix} & 0 & 0 \\ 0 & \delta^2 V_\mu & 0 \\ 0 & 0 & \delta^2 K_\mu \end{bmatrix} \end{array}$$

and

$$\Omega_{z_e} = \begin{array}{c} \begin{array}{ccc} \qquad\mathcal{S}_{\mathrm{RIG}} \qquad & \qquad \mathcal{W}_{\mathrm{INT}} \qquad & \qquad \mathcal{W}^*_{\mathrm{INT}} \end{array} \\ \left[\begin{array}{ccc} \begin{bmatrix} \text{Coadjoint orbit} \\ \text{symplectic form} \end{bmatrix} & \begin{bmatrix} \text{Internal-Rigid} \\ \text{coupling} \end{bmatrix} & 0 \\ -\begin{bmatrix} \text{Internal-Rigid} \\ \text{coupling} \end{bmatrix} & \multicolumn{2}{c}{\begin{bmatrix} \text{Canonical symplectic} \\ \text{form plus a} \\ \text{magnetic term} \end{bmatrix}} \\ 0 & & \end{array} \right] \end{array}$$

Here and below, we write $\xi = \xi_e$ and $\mu = \mu_e$ for simplicity. The terms appearing in the formula for Ω_{z_e} will be explained below. We illustrate some steps in the proof of the theorem as follows.

4.2 Lemma $\delta^2 H_\xi(z_e)\cdot(\Delta z, \delta z) = 0$ *for all* $\Delta z \in \mathcal{S}_{\mathrm{RIG}}$ *and* $\delta z \in \mathcal{S}_{\mathrm{INT}}$.

To prove this, recall that H_ξ can be written (see equation (10) of §1):

$$H_\xi(z) = V(q) + \tfrac{1}{2} J(z)\cdot \mathbb{I}^{-1}(q) J(z) - [J(z) - \mu]\cdot\xi + K(\Sigma(z)). \tag{1}$$

Given $\mu \in \mathfrak{g}^*$, define the functions $\rho : T^*Q \to \mathfrak{g}$ and $\mathcal{Q} : T^*Q \to \mathbb{R}$ by

$$\rho(z) = \tfrac{1}{2}\mathbb{I}^{-1}(q)(J(z) + \mu) - \xi \tag{2}$$

and

$$\mathcal{Q}(z) = \langle J(z) - \mu, \rho(z)\rangle . \tag{3}$$

We can now rewrite H_ξ in the form

$$H_\xi(z) = K(\Sigma(z)) + \mathcal{Q}(z) + V_\mu(q). \tag{4}$$

This regrouping of H_ξ is convenient because, as we saw before, at a critical point $z_e \in J^{-1}(\mu)$, each of the three terms has zero first variation. We shall show that the second variations of these terms restricted to variations of the form $(\Delta z, \delta z) \in \mathcal{S}_{\mathrm{RIG}} \times \mathcal{S}_{\mathrm{INT}}$ equal zero. The first variation of the term $K(\Sigma(z)) = \frac{1}{2}\| p - \alpha_z \|^2$ equals zero at a relative equilibrium, since $p_e = \alpha_{z_e}$. The first variation of the term $\mathcal{Q}(z)$ is

$$\mathbf{D}\mathcal{Q}(z_e)\cdot\delta z = \langle \mathbf{D}\mathbf{J}(z_e)\cdot\delta z, \rho(z_e)\rangle + \langle \mathbf{J}(z_e) - \mu, \mathbf{D}\rho(z_e)\cdot\delta z\rangle = 0,$$

since $\mathbf{J}(z_e) = \mu = \mathbb{I}(q_e)\xi$ implies $\rho(z_e) = 0$. Thus, $\mathbf{D}H_\xi(z_e) = 0$ if and only if $\mathbf{D}V_\mu(q_e) = 0$.

By construction, $T_{z_e}\Sigma\cdot\Delta z = (\Delta q, 0)$ for $\Delta z \in \mathcal{S}_{\mathrm{RIG}}$; hence

$$\mathbf{D}^2(K\circ\Sigma)(z_e)(\Delta z, \delta z) = \langle T_{z_e}\Sigma\cdot\Delta z, T_{z_e}\Sigma\cdot\delta z\rangle = 0$$

for $\Delta z \in \mathcal{S}_{\mathrm{RIG}}$ and arbitrary δz. Next, we show that $\mathbf{D}^2\mathcal{Q}(z_e)\,|_{\mathcal{S}\times\mathcal{S}} = 0$. Let $\delta z, \widetilde{\delta z} \in \mathcal{S}$; then

$$\begin{aligned}\mathbf{D}^2\mathcal{Q}(z_e)(\delta z, \widetilde{\delta z}) &= \langle \mathbf{J}(z_e) - \mu, \mathbf{D}^2\rho(z_e)(\delta z, \widetilde{\delta z})\rangle + \langle \mathbf{D}\mathbf{J}(z_e)\cdot\delta z, \mathbf{D}\rho(z_e)\cdot\widetilde{\delta z}\rangle \\ &\quad + \langle \mathbf{D}\mathbf{J}(z_e)\cdot\widetilde{\delta z}, \mathbf{D}\rho(z_e)\cdot\delta z\rangle + \langle \mathbf{D}^2\mathbf{J}(z_e)(\delta z, \widetilde{\delta z}), \rho(z_e)\rangle = 0,\end{aligned}$$

since $\mathbf{J}(z_e) = \mu$, $\mathbf{D}\mathbf{J}(z_e)\,|_{\mathcal{S}} = 0$, and $\rho(z_e) = 0$.

Finally, we show that $\mathbf{D}^2V_\mu(q_e)(\Delta q, \delta q) = 0$ for $\Delta q \in \mathcal{V}_{\mathrm{RIG}}$ and $\delta q \in \mathcal{V}_{\mathrm{INT}}$. Let $\Delta q = \eta_Q(q_e)$ and $\zeta = (\mathbf{D}\mathbb{I}(q_e)^{-1}\cdot\delta q)\mu$. Since $\mathbf{D}\mathcal{V}_\mu(q_e) = 0$, we have

$$\begin{aligned}\mathbf{D}^2\mathcal{V}_\mu(q_e)(\Delta q, \delta q) &= \mathbf{D}(\mathbf{D}\mathcal{V}_\mu(q_e)\cdot\Delta q)\cdot\delta q = \mathbf{D}(\mathrm{ad}^*_\eta\mu\cdot\mathbb{I}(q_e)^{-1}\mu)\cdot\delta q \\ &= (\mathrm{ad}^*_\eta\mu\cdot(\mathbf{D}\mathbb{I}(q_e)^{-1}\cdot\delta q)\mu = \mathrm{ad}^*_\eta\mu\cdot\zeta = 0,\end{aligned}$$

since $\mathrm{ad}^*_\eta\mu \in (\mathfrak{g}_{\mu_e})^A$ and $\delta q \in \mathcal{V}_{\mathrm{INT}}$ implies $\zeta \in \mathfrak{g}_\mu$. ∎

We refer the reader to Simo, Lewis and Marsden [1990] for the calculation of $\delta^2 H_\xi$ on the remaining blocks.

Now we turn to the symplectic form $\Omega_{z_e} = \Omega(z_e)$. In Simo, Lewis, and Marsden [1990] it is shown that for $\Delta q \in \mathcal{V}_{\mathrm{RIG}}$, $\Delta p = T_{q_e}\alpha_\mu\cdot\Delta q$ is given by $\Delta p = \mathcal{V}(\mathbb{F}L\cdot\zeta_Q(q_e)) - T^*_{z_e}\eta_Q\cdot p_e$, where

$$\Delta q = \eta_Q(q_e) \text{ and } \zeta = \mathbb{I}(q_e)^{-1}\mathrm{ad}^*_\eta\mu\,.$$

Using this notation, we have

4.3 Lemma

For $\delta z \in T_{z_e}P$,

$$\Omega(z_e)(\Delta z, \delta z) = (\mathbf{D}\mathbf{J}(z_e)\cdot\delta z)\cdot\eta - \langle\langle \zeta_Q(q_e), \delta q\rangle\rangle\,.$$

ii *For* $\delta z \in \mathcal{S}_{\mathrm{INT}}$ *, the rigid internal coupling terms are*

$$\Omega(z_e)(\Delta z, \delta z) = -\langle\langle \zeta_Q(q_e), \delta q\rangle\rangle = -\langle \mu_e, [\eta, \alpha(\delta q)]\rangle .$$

iii *For* $\Delta z = T_{z_e}Z_\mu \cdot \eta_Q(q_e)$ and $\widetilde{\Delta z} = T_{z_e}Z_\mu \cdot \tilde{\eta}_Q(q_e) \in \mathcal{S}_{\mathrm{RIG}}$ *, the coadjoint orbit, or Lie-Poisson terms are*

$$\Omega(z_e)(\Delta z, \widetilde{\Delta z}) = -\mu_e \cdot [\eta, \tilde{\eta}] .$$

iv *For given variations* $\delta z = T_{q_e}\alpha_\mu \cdot \delta q$ and $\widetilde{\delta z} = T_{q_e}\alpha_\mu \cdot \widetilde{\delta z} \in \mathcal{W}_{\mathrm{INT}}$, *the magnetic terms are*

$$\Omega(z_e)(\delta z, \widetilde{\delta z}) = \mathbf{d}\alpha_\mu(\widetilde{\delta q}, \delta q) = \mathbf{d}\alpha_\xi(\widetilde{\delta q}, \delta q),$$

where $\alpha_\xi : Q \to T^*Q$ *is the one form given by* $\alpha_\xi(q) = \mathbb{F}L(\xi_Q(q))$.

v *For* $\delta z = T_{q_e}\alpha_\mu \cdot \delta q \in \mathcal{W}_{\mathrm{INT}}$, *and* $\widetilde{\delta z} = (0, \widetilde{\delta z}) \in \mathcal{W}^*_{\mathrm{INT}}$, *the canonical terms are*

$$\Omega(z_e)(\delta z, \widetilde{\delta z}) = \langle \widetilde{\delta z}, \delta q\rangle .$$

vi *For given variations* $\delta z, \widetilde{\delta z} \in \mathcal{W}^*_{\mathrm{INT}}$, *we have* $\Omega(z_e)(\delta z, \widetilde{\delta z}) = 0$.

Proof

$$\begin{aligned}\Omega(z_e)(\Delta z, \delta z) &= \langle \delta p, \Delta q\rangle - \langle \Delta p, \delta q\rangle \\ &= \langle \delta p, \eta_Q(q_e)\rangle + \langle p_e, T_{q_e}\eta_Q \cdot \delta q\rangle - \langle\langle \zeta_Q(q_e), \delta q\rangle\rangle \\ &= (\mathbf{D}\mathbf{J}(z_e)\cdot \delta z)\cdot \eta - \langle\langle \zeta_Q(q_e), \delta q\rangle\rangle .\end{aligned}$$

Hence **i** holds. If $\delta z \in T_{q_e}\mathbf{J}^{-1}(\mu_e)$, then $\mathbf{D}\mathbf{J}(z_e)\cdot \delta z = 0$, so the first equality in **ii** holds as well. Also,

$$\langle\langle \zeta_Q(q_e), \delta q\rangle\rangle = \langle \alpha_{\mathrm{ad}^*_\eta \mu}(q_e), \delta q\rangle = \langle \mathrm{ad}^*_\eta \mu, \alpha(\delta q)\rangle = \langle \mu, [\eta, \alpha(\delta q)]\rangle .$$

Thus we have proved **ii**. Statement **iii** follows from **ii** and the fact that $\alpha(\eta_Q(q_e)) = \eta$. To show **iv**, we check that

$$\begin{aligned}\Omega(z_e)(\delta z, \widetilde{\delta q}) &= \langle \mathbf{D}\alpha_\mu(q_e)\cdot \widetilde{\delta q}, \delta q\rangle - \langle \mathbf{D}\alpha_\mu(q_e)\cdot \delta q, \widetilde{\delta q}\rangle \\ &= \mathbf{D}(\langle \alpha_\mu(q_e), \delta q\rangle)\cdot \widetilde{\delta q}, - \mathbf{D}(\langle \alpha_\mu(q_e), \widetilde{\delta q}\rangle)\cdot \delta q - \alpha_\mu(q_e)\cdot \pounds_{\delta q}\widetilde{\delta q}\end{aligned}$$

$$= \mathbf{d}\alpha_{\mu_e}(\widetilde{\delta q}, \delta q),$$

where $\pounds$ is the Lie derivative—in this calculation, δq and $\widetilde{\delta q}$ are temporarily extended to vector fields. On the other hand,

$$\begin{aligned}\langle \mathbf{D}\alpha_{\mu_e}(q_e)\cdot\widetilde{\delta q}, \delta q\rangle &= \langle \mathbf{D}\alpha_\xi(q_e)\cdot\widetilde{\delta q}, \delta q\rangle + \langle\langle [\mathbf{D}\mathbb{I}^{-1}(q_e)\cdot\widetilde{\delta q})\cdot\mu_e]_Q(q_e), \delta q\rangle\rangle \\ &= \langle \mathbf{D}\alpha_\xi(q_e)\cdot\widetilde{\delta q}, \delta q\rangle,\end{aligned}$$

since $\widetilde{\delta q} \in \mathcal{V}_{\mathrm{INT}}$ implies $v = (\mathbf{D}\mathbb{I}^{-1}(q_e)\cdot\widetilde{\delta q})\mu_e \in \mathfrak{g}_{\mu_e}$ and so $\langle\langle v_Q(q_e), \delta q\rangle\rangle = 0$ by defintion of $\mathcal{V}$. Thus, we can replace α_{μ_e} by α_ξ to obtain $\Omega(z_e)(\delta z, \widetilde{\delta z}) = \mathbf{d}\alpha_\xi(\widetilde{\delta q}, \delta q)$, which completes the proof of **iv**. Parts **v** and **vi** follow from the definition of the canonical symplectic form. ■

Notice that if $\delta^2 H_\xi$ is to be definite, the sign must be positive since the term $\delta^2 K_\mu$ is always positive unless it is absent altogether. These results lead to:

4.4 Reduced Energy-Momentum method *Assume that the* $\delta^2 K_\mu$ *block is nontrivial (in the finite dimensional case, this means that* $\dim Q \neq \dim G$).

Necessary and sufficient conditions for formal stability are that the following two conditions hold:

i *The Arnold form* $\mathcal{A}_\mu : \mathfrak{g}_\mu^\perp \times \mathfrak{g}_\mu^\perp \to \mathbb{R}$ *be positive definite*

and

ii *The second variation* $\delta^2 V_\mu(q_e)$ *restricted to* $\mathcal{V}_{\mathrm{INT}}$ *be positive definite.*

In the case that the block $\delta^2 K_\mu$ is trivial, then necessary and sufficient conditions for formal stability are that the Arnold form be definite, either positive or negative, a case that was treated already by Arnold [1966]. If the group G is abelian, then it follows from work of Smale [1970] (see also Abraham and Marsden [1978]) that the condition for formal stability is just positive definiteness of $\delta^2 V_\mu$ since then the Arnold block is trivial and the system reduces to a simple mechanical one (with a magnetic term) on $T^*(Q/G)$. Thus, *the reduced energy-momentum method can be thought of as a synthesis of the work of Arnold and Smale.*

§5 *The Normalizing Connection*

We begin by describing connections on the level of configuration spaces. These are then lifted via the mechancal connection to phase space as in §3. We consider the following sequence:

$$Q \xrightarrow{\varphi_\mu} Q/G_\mu \xrightarrow{\sigma_\mu} Q/G = S$$

where φ_μ and σ_μ are the natural projections. Let γ_μ be the connection on $Q \to Q/G_\mu$ whose horizontal space is metric orthogonal to the G_μ orbits. This is the mechanical connection regarded on the G_μ bundle $Q \to Q/G_\mu$

At a particular $q \in Q$ (fixed in the discussion), the corresponding space $\mathcal{V}$ is

$$\mathcal{V} = \mathrm{hor}(\gamma_\mu) \tag{1}$$

Thus, $\mathcal{V}$ defines γ_μ and vice versa.

5.1 Proposition $\mathrm{hor}_{\gamma_\mu}(\mathrm{vert}\,\sigma_\mu) = \mathcal{V}_{\mathrm{RIG}}$, *where* $\mathrm{hor}_{\gamma_\mu}$ *is the horizontal lift with respect to* γ_μ *and* $\mathrm{vert}\,\sigma_\mu$ *is the vertical space of* σ_μ:

$$\mathrm{vert}\,\sigma_\mu = \{ w \in T_{\varphi_\mu(q)}(Q/G_\mu) \mid T\sigma_\mu \cdot w = 0 \} \tag{2}$$

Proof Let $\eta_Q(q) \in \mathcal{V}_{\mathrm{RIG}}$, so $\eta \in \mathfrak{g}_\mu^\perp$. Thus $T\sigma_\mu \cdot T\varphi_\mu \cdot \eta_Q(q) = T(\sigma_\mu \circ \varphi_\mu) \cdot \eta_Q(q)$. This is zero because $\sigma_\mu \circ \varphi_\mu$ projects $\zeta_Q(q)$ to zero for all $\zeta \in \mathfrak{g}$. Since $\mathcal{V}_{\mathrm{RIG}} \subset \mathcal{V}$, $\mathcal{V}_{\mathrm{RIG}}$ is horizontal, so we have proved

$$\mathcal{V}_{\mathrm{RIG}} \subset \mathrm{hor}_{\gamma_\mu}(\mathrm{vert}\,\sigma_\mu)$$

We prove the inclusion $\supset$ as follows. If $w_q \in \mathrm{hor}_{\gamma_\mu}(\mathrm{vert}\,\sigma_\mu)$, then $T\sigma_\mu \cdot T\varphi_\mu \cdot w_q = 0$, so $w_q = \eta_Q(q)$ for some $\eta \in \mathfrak{g}$. But if $\eta_Q(q)$ is γ_μ-horizontal, then $\eta \in \mathfrak{g}_\mu^\perp$, so $\mathrm{hor}_{\gamma_\mu}(\mathrm{vert}\,\sigma_\mu) \subset \mathcal{V}_{\mathrm{RIG}}$. ■

Now consider the split

$$\mathcal{V} = \mathcal{V}_{\mathrm{RIG}} \oplus \mathcal{V}_{\mathrm{INT}} \subset T_qQ$$

In view of the propositon, we can identify $\mathcal{V}$ with $T_{[q]}(Q/G_\mu)$ and $\mathcal{V}_{\mathrm{RIG}}$ with the vertical space. Thus, $\mathcal{V}_{\mathrm{INT}}$ *defines a connection, say* δ_μ on the bundle

$$Q/G_\mu \xrightarrow{\sigma_\mu} Q/G = S$$

The connections γ_μ and δ_μ induce connections on the other bundles in the cotangent bundle reduction theorem. (See the figure in §2). For example, the split $\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$ may be viewed as the vertical-horizontal split of a connection on the bundle

$$P_O \to T^*S$$

with fiber the coadjoint orbit O. Similarly, the split $\mathcal{S}_{INT} = \mathcal{W}_{INT} \oplus \mathcal{W}^*_{INT}$ is a connection on the bundle $T^*S \to S$ pulled up to P_O by the projection $P_O \to T^*S$, as in formulas (7-10) of §3.

Another question is interesting here. We saw that σ_μ is a principal bundle connection on $Q \to Q/G_\mu$. In what sense can δ_μ be viewed as a principal connection? There are two ways to answer this question. The ***first method*** allows μ to vary while the ***second method*** fixes μ. Turning to the first method, define the manifold

$$\tilde{Q} = \{(q, \nu) \in Q \times \mathfrak{g}^* \mid \mathbb{I}(q)^{-1}\nu \in \mathfrak{g}_\nu\} \tag{3}$$

Under our assumptions, $\mathbb{I}(q)$ is invertible, and this is enough to show that $\tilde{Q}$ is a manifold, as a transversality argument shows. In fact, the map $(q, \nu) \mapsto \mathbb{I}(q)^{-1}\nu$ is transversal to $\mathfrak{g}_\nu$ and so $\tilde{Q}$ is a manifold of dimension that of $Q \times \mathfrak{g}_\nu$.

Remark Enlarging the configuration space from Q to $Q \times G$ where we imagine the system together with an orthonormal frame, is a natural procedure, as in Krishnaprasad and Marsden [1987]. The conditions $\mathbb{I}(q)^{-1}\mu \in \mathfrak{g}_\mu$ on $Q \times \mathfrak{g}^*$ and $J(z) - \mu = 0$, on $T^*(Q \times G)$, may be viewed as "locking devices" that lock the orthonormal frame to the structure. Reduction by G produces $T^*Q \times \mathfrak{g}^*$ and μ is eliminated by the condition $J(z) = \mu$ (*i.e.*, the momentum map for the total action of G on $T^*(Q \times G)$ is zero.)

Let G act on $\tilde{Q}$ by the product of the action on Q with the coadjoint action on $\mathfrak{g}^*$. By equivariance properties of $\mathbb{I}$ noted earlier, this action preserves $\tilde{Q}$. Consider the principal bundle

$$\tilde{Q} \to \tilde{Q}/G \tag{4}$$

and note that the quotient space $\tilde{Q}/G$ has the dimension of $S \times \mathfrak{g}_\mu$.

5.2 Proposition *The choice of horizontal space* $\mathcal{V}_{\mathrm{INT}} \oplus T_\mu \mathcal{O} \subset T_{(q,\mu)}\tilde{Q}$ *defines a principal* G-*connection on the bundle* (4).

The connection form ε associated with this connection can be described as follows. Define the map $\beta_{(q,v)} : T_{(q,v)}\tilde{Q} \to \mathfrak{g}_v$ by $(\delta q, \delta v) \mapsto \mathbb{P}_v(\alpha(\delta q))$, where $\mathbb{P}_v$ is the orthogonal projection onto $\mathfrak{g}_v$. Also, define the map $\beta^{\perp}_{(q,v)} : T_{(q,v)}\tilde{Q} \to \mathfrak{g}_v^{\perp}$ by

$$\beta^{\perp}_{(q,v)}(\delta q, \delta v) = \chi^{-1}_{(q,v)}(\mathbb{P}^{\perp}_{(q,v)}\{[D\mathbb{I}(q)^{-1}\cdot \delta q]\cdot v\}) = \chi^{-1}_{(q,v)}(\mathbb{P}^{\perp}_{(q,v)}\{\mathbb{I}(q)^{-1}\delta v\})$$

where $\mathbb{P}^{\perp}_{(q,v)}$ is the projection onto $\mathfrak{g}_v^{\perp}$. Now set

$$\varepsilon(\delta q, \delta v) = \beta_{(q,v)}(\delta q, \delta v) + \beta^{\perp}_{(q,v)}(\delta q, \delta v) \tag{5}$$

It is straight forward to check that the form ε is the connection one form for the connection in **5.2** and that it is a G-connection.

This connection can be lifted up to $\tilde{P} = \{z \in T^*Q \mid \mathbb{I}(q)^{-1}J(z) \in \mathfrak{g}_{J(z)}\}$ as before. Note that the inverse image of a point $(q, \mu) \in \tilde{Q}$ under the projection of $\tilde{P}$ to $\tilde{Q}$ is $J^{-1}(\mu) \cap T^*_q Q$.

The *second method* proceeds as follows. Here we keep μ fixed and define the manifold

$$Q_\mu = \{q \in Q \mid \mathbb{I}(q)^{-1}\mu \in \mathfrak{g}_\mu\}. \tag{6}$$

Note that Q_μ is a fiber of $\tilde{Q}$ obtained by fixing v at the value μ. The tangent space to Q_μ is given by linearizing the defining condition; one finds that

$$T_q Q_\mu = \{\delta q \in T_q Q \mid \eta\cdot(D\mathbb{I}(q)\cdot\delta q)\xi = 0 \text{ for all } \eta \in \mathfrak{g}_\mu^{\perp}\} \equiv \mathcal{V}. \tag{7}$$

From this and the fact that G_μ acts on Q_μ, we see that *the tangent to the quotient space* Q_μ/G_μ *is isomorphic to* $\mathcal{V}_{\mathrm{INT}}$. Thus, if we take $\mathcal{V}$ to be given by (7) and regard it as being isomorphic to the original definition ((1a) of §3), then the split $\mathcal{V} = \mathcal{V}_{\mathrm{RIG}} \oplus \mathcal{V}_{\mathrm{INT}}$ is just the horizontal-vertical split of the principal connection on the bundle $Q_\mu \to Q_\mu/G_\mu$ defined by declaring the horizontal space to be the metric orthogonal to the G_μ-orbits. In this viewpoint, one replaces the sequence $Q \to Q/G_\mu \to Q/G$ and its corresponding connections with the sequence $Q \supset Q_\mu \to Q_\mu/G_\mu$. Then one lifts the connections to phase spaces as before and blends them with the connection corresponding to the splitting $\mathcal{S}_{\mathrm{INT}} = \mathcal{W}_{\mathrm{INT}} \oplus \mathcal{W}^*_{\mathrm{INT}}$.

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